Action of Raising and Lowering Operators on Harmonic Oscillator Eigenstates

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Exercise:

We show that the raising and lowering operators

$$a = \sqrt{\frac{1}{2m\hbar\omega}}(m\omega x + ip) \quad , \quad a^\dagger = \sqrt{\frac{1}{2m\hbar\omega}}(m\omega x - ip)$$

take harmonic oscillator eigenstates to other eigenstates of higher or lower energy, respectively.

Solution: Firstly, given the definitions above, we have that the simple harmonic oscillator Hamiltonian is given by

$$H = \hbar\omega(aa^\dagger - \frac{1}{2}) = \hbar\omega(a^\dagger a + \frac{1}{2}).$$

Suppose we have an energy eigenstate $|E\rangle$, such that $H|E\rangle = E|E\rangle$.

We consider the action of the raising operator on this state.

$$Ha^\dagger|E\rangle = \hbar\omega(a^\dagger a + \frac{1}{2})a^\dagger|E\rangle = \hbar\omega a^\dagger(aa^\dagger + \frac{1}{2})|E\rangle =$$
$$\hbar\omega a^\dagger(aa^\dagger - \frac{1}{2} + 1)|E\rangle = a^\dagger(H + \hbar\omega)|E\rangle = a^\dagger(E + \hbar\omega)|E\rangle = (E + \hbar\omega)a^\dagger|E\rangle \implies$$
$$\boxed{Ha^\dagger|E\rangle = (E + \hbar\omega)a^\dagger|E\rangle.}$$

Thus, the raising operator takes an eigenstate of energy E to another eigenstate of energy $E + \hbar\omega$.

Next, we consider the action of the lowering operator on this state.

$$Ha|E\rangle = \hbar\omega(aa^\dagger - \frac{1}{2})a|E\rangle = \hbar\omega a(a^\dagger a + \frac{1}{2} - 1)|E\rangle =$$

$$a(H - \hbar\omega)|E\rangle = a(E - \hbar\omega)|E\rangle = (E - \hbar\omega)a|E\rangle \implies$$
$$\boxed{Ha|E\rangle = (E - \hbar\omega)a|E\rangle.}$$

Thus, the lowering operator takes an eigenstate of energy E to another eigenstate of energy $E - \hbar\omega$.