We derive the Bose-Einstein distribution, which describes the average number of particles occupying a given state in a Bose gas.

We consider a Bose gas of identical, noninteracting bosons. Suppose that the particle number is not conserved, so that the energy of state $|r\rangle$ is given by

$$n_r(E_r - \mu)$$

Where n_r is the number of particles in state $|r\rangle$, E_r is the energy for a single particle in state $|r\rangle$, and μ is the chemical potential. Since the particles are identical, the system is completely specified by the number of particles in each state. It follows that the grand partition function is given by

$$\mathcal{Z} = \prod_r \mathcal{Z}_r = \prod_r \left(\sum_{n_r=0}^{\infty} e^{-\beta n_r(E_r-\mu)} \right) = \prod_r \frac{1}{1-e^{-\beta(E_r-\mu)}}$$

where we have assumed $E_r \geq 0, \mu < 0$ to ensure convergence.

Generally, the grand partition function is given by

$$\mathcal{Z} = \sum_n e^{-\beta(E_n - \mu N_n)} \Rightarrow \frac{1}{\beta}\frac{\partial}{\partial \mu} \log \mathcal{Z} =$$

$$\frac{1}{\beta} \cdot \frac{1}{\mathcal{Z}} \sum_n e^{-\beta(E_n-\mu N_n)} \cdot \beta N_n = \frac{1}{\mathcal{Z}} \sum_n N_n e^{-\beta(E_n-\mu N_n)}$$

$= \langle N \rangle$, the average number of particles in the system.

Applying this formula to our grand partition function gives us

$$\langle N \rangle = \frac{1}{\beta} \frac{\partial}{\partial \mu} \log \mathcal{Z} = -\frac{1}{\beta} \sum_r \frac{\partial}{\partial \mu} \log\left(1 - e^{-\beta(E_r - \mu)}\right)$$

$$= -\frac{1}{\beta} \sum_r \frac{-e^{-\beta(E_r-\mu)}\beta}{1-e^{-\beta(E_r-\mu)}} = \sum_r \frac{1}{e^{\beta(E_r-\mu)}-1} \equiv \sum_r \langle n_r \rangle,$$

where n_r is the expected number of particles in state $|r\rangle$. Thus, the Bose-Einstein distribution is given by

$$\boxed{\langle n_r \rangle = \frac{1}{e^{\beta(E_r-\mu)}-1}}$$