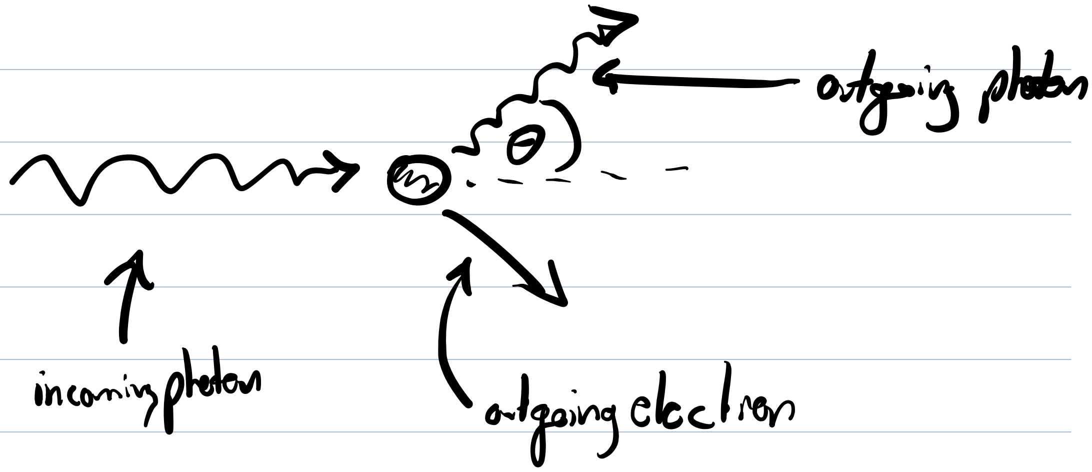
Exercise We derive the Compton scattering formula.

(Source: Wikipedia)

We consider the inelastic scattering between a photon and an electron which is initially at rest. The inelasticity implies that the photon transfers momentum to the electron, and we allow for the possibility of the electron being accelerated to relativistic speeds



The 4-momentum will be conserved during this collision

$$p_e^\mu + p_\gamma^\mu = p_{e'}^\mu + p_{\gamma'}^\mu .$$

Separating into components, we have

$$\begin{cases} m_e c^2 + \hbar\omega = E_{e'} + \hbar\omega' \\ \vec{p}_\gamma = \vec{p}_{\gamma'} + \vec{p}_{e'} \end{cases} \implies$$

$$\begin{cases} m_e c^2 + \hbar(\omega - \omega') = \sqrt{m_e^2 c^4 + p_e'^2 c^2} \\ c^2 (p_\gamma^2 - 2\vec{p}_\gamma \cdot \vec{p}_\gamma' + p_\gamma'^2) = p_{e'}^2 c^2 , \end{cases}$$

Where we have used the relativistic dispersion relation $E^2 = p^2c^2 + m^2c^4$. Now, expanding and using $E = pc$, we have

$$m_e^2 c^4 + 2m_e c^2 \hbar(\omega - \omega') + \hbar^2(\omega - \omega')^2 - m_e^2 c^4 = p_{e'}^2 c^2$$

$$\hbar^2(\omega^2 - 2\omega\omega' \cos\theta + \omega'^2) = p_{e'}^2 c^2 ,$$

and equating those two, we have

$$2m_e c^2 \hbar(\omega - \omega') + \hbar^2(\omega^2 - 2\omega\omega' + \omega'^2) =$$

$$\hbar^2(\omega^2 - 2\omega\omega' \cos\theta + \omega'^2) \iff$$

$$2m_e c^2 \hbar(\omega - \omega') = 2\hbar^2 \omega\omega'(1 - \cos\theta) \iff$$

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{\hbar}{m_e c^2}(1 - \cos\theta) \iff$$

$$\boxed{\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)}$$

This is the Compton scattering formula, and $h/m_e c$ is the Compton wavelength.