Conservation of Probability Current

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Exercise:

We demonstrate the conservation of probability current in quantum mechanics.

Solution:

The probability current is defined as

$$J(x,t) = \frac{i\hbar}{2m}\Big(\frac{\partial \Psi^*}{\partial x}\Psi - \Psi^*\frac{\partial \Psi}{\partial x}\Big).$$

We consider the probability that a particle is found between a and b :

$$P_{ab} = \int_a^b |\Psi|^2 dx.$$

Then, this quantity changes in time as

$$\dot{P}_{ab} = \int_a^b \partial_t |\Psi|^2 dx = \int_a^b \Big(\dot{\Psi}^*\Psi + \Psi^*\dot{\Psi}\Big)dx =$$
$$\int_a^b \frac{i}{\hbar}\Big[\Big(-\frac{\hbar^2}{2m}\Psi^*_{xx}\Psi + V|\Psi|^2\Big) - \Big(-\frac{\hbar^2}{2m}\Psi^*\Psi_{xx} + V|\Psi|^2\Big)\Big]dx =$$
$$\frac{i\hbar}{2m}\int_a^b \Big(\Psi^*\Psi_{xx} - \Psi^*_{xx}\Psi\Big)dx = \frac{i\hbar}{2m}\Big(\Psi^*\Psi_x - \Psi^*_x\Psi\Big)\Big|_a^b =$$
$$-J(x,t)\Big|_a^b = J(x,t)\Big|_b^a = \boxed{J(a,t) - J(b,t).}$$

Thus, in quantum mechanics, the change in probability in some region of space equals the net flux of probability current at the boundary.