Exercise | We derive Cramer's rule for solving two linear equations in two unknowns.

(Source: Boas, 3.3)

We consider a system of two linear equations in two unknowns:

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

We solve for x and y. First, we multiply equation 1 by b_2 and equation 2 by b_1. Then, we have

$$b_2 a_1 x + b_2 b_1 y = b_2 c_1$$
$$b_1 a_2 x + b_1 b_2 y = b_1 c_2 \Rightarrow$$

$$(a_1 b_2 - a_2 b_1) x = c_1 b_2 - c_2 b_1 \Rightarrow$$

$$x = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Similarly, if we multiply equation 1 by a_2 and equation

2 by a_1, we get

$$a_2 a_1 x + a_2 b_1 y = a_2 c_1$$
$$a_1 a_2 x + a_1 b_2 y = a_1 c_2 \implies$$

$$(a_1 b_2 - a_2 b_1) y = a_1 c_2 - a_2 c_1 \implies$$

$$y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Thus, provided that the determinant of the coefficient matrix doesn't vanish, the system of linear equations is solved by

$$\boxed{x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}}$$