We compute the density of states for a quantum particle in a 3D box with periodic boundary conditions.

We previously derived that the energy of this state is given by

$$E = \frac{\hbar^2 \vec{k}^2}{2m} = \frac{\hbar^2}{2m}\left[\left(\frac{2\pi n_x}{L}\right)^2 + \left(\frac{2\pi n_y}{L}\right)^2 + \left(\frac{2\pi n_z}{L}\right)^2\right],$$

where $\vec{k}$ is the wavevector for the state $\Psi = 1/\sqrt{V}\, \exp(i\vec{k}\cdot\vec{x})$, and $n_i \in \mathbb{Z}$ is a discretization forced on the wave vectors due to periodic boundary conditions.

The density of states is defined to be the number of energy eigenstates between E and $E + dE$. Given that

$$E = \frac{2\pi^2\hbar^2}{mL^2} n^2 ,$$

the region between E and $E+dE$ in n-space will form a spherical shell containing $4\pi n(E)^2\, dn(E)$ points. From our expression above, we have

$$n^2 = \frac{mL^2 E}{2\pi^2\hbar^2}, \quad dE = \frac{4\pi^2\hbar^2}{mL^2}\, n\, dn$$

Combining these, we have

$$4\pi n^2 dn = \cancel{4\pi} \cdot \frac{mL^2 E}{2\pi^2 \hbar^2} \cdot \frac{m\cancel{L^2}\, dE}{\cancel{4\pi^2}\,\cancel{\hbar^2}} \left(\frac{2\cancel{\pi^2}\,\cancel{\hbar^2}}{m\cancel{L^2} E} \right)^{1/2} =$$

$$\frac{m^{3/2} L^3}{\sqrt{2}\,\pi^2 \hbar^3} \sqrt{E}\, dE = \boxed{\frac{m^{3/2} V}{\sqrt{2}\,\pi^2 \hbar^3} \sqrt{E}\, dE = g(E)\, dE}$$