Derivation of the Momentum Operator in Quantum Mechanics

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We derive the momentum operator in quantum mechanics.

Recall that the expectation value of the position operator is given by

$$\langle x\rangle = \int x|\Psi(x,t)|^2 dx.$$

This quantity evolves in time as

$$\frac{d}{dt}\langle x\rangle = \frac{d}{dt}\int x|\Psi(x,t)|^2 dx = \int \frac{\partial}{\partial t}\Big(x|\Psi(x,t)|^2\Big)dx = \int \Big(\dot{x}|\Psi(x,t)|^2 + x\frac{\partial}{\partial t}|\Psi(x,t)|^2\Big)dx =$$
$$\int x\frac{\partial}{\partial t}|\Psi(x,t)|^2 dx = \int x\Big(\dot{\Psi}^*\Psi + \Psi^*\dot{\Psi}\Big)dx.$$

Plugging into the Schrödinger equation, we have

$$\frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\Big(-\frac{\hbar^2}{2m}\Big)\int x\Big(\Psi^*_{xx}\Psi - \Psi^*\Psi_{xx}\Big)dx = \frac{i}{\hbar}\Big(-\frac{\hbar^2}{2m}\Big)\int x\partial_x\Big(\Psi^*_x\Psi - \Psi^*\Psi_x\Big)dx =$$
$$\frac{i\hbar}{2m}\int\Big(\Psi^*_x\Psi - \Psi^*\Psi_x\Big)dx = \frac{i\hbar}{2m}\int -2\Psi^*\Psi_x dx =$$
$$\boxed{\frac{1}{m}\int \Psi^*\Big(-i\hbar\partial_x\Big)\Psi dx \equiv \frac{1}{m}\langle p\rangle.}$$