Exercise: We compute $\frac{d}{dx}\int_a^{f(x)} g(t)\,dt$.

Let G be the antiderivative of g, such that

$$\int_a^b g(t)\,dt = G(b) - G(a).$$

Then, $G(x) = \int_a^x g(t)\,dt + G(a)$.

We can now differentiate G:

$$G'(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[G(x+\varepsilon) - G(x)\right] =$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\int_a^{x+\varepsilon} g(t)\,dt - \int_a^x g(t)\,dt\right] =$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\int_a^{x+\varepsilon} g(t)\,dt + \int_x^a g(t)\,dt\right] =$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_x^{x+\varepsilon} g(t)\,dt \approx \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} g(x) \cdot \varepsilon = g(x).$$

This is the fundamental theorem of calculus.

Now, we compute

chain rule

$$\frac{d}{dx} \int_a^{f(x)} g(t)\,dt = \frac{d}{dx} G(f(x)) = G'(f(x)) \cdot f'(x)$$

$$= g(f(x)) \cdot f'(x) \quad \rightarrow$$

$$\boxed{\frac{d}{dx} \int_a^{f(x)} g(t)\,dt = g(f(x)) \cdot f'(x)}$$