Ehrenfest's Theorem

Matt Kafker

We derive Ehrenfest's Theorem, which says that expectation values in quantum mechanics obey classical equations of motion.

To demonstrate this, we compute $d\langle p\rangle/dt$.

$$\begin{gathered}
\frac{d\langle p\rangle}{dt} = \frac{d}{dt}\int \Psi^*\Big(-i\hbar\partial_x\Big)\Psi dx \implies \\
\frac{i}{\hbar}\frac{d\langle p\rangle}{dt} = \int \partial_t\Big(\Psi^*\Psi_x\Big)dx = \int\Big(\dot{\Psi}^*\Psi_x + \Psi^*\dot{\Psi}_x\Big)dx = \\
\frac{i}{\hbar}\int\Big[-\frac{\hbar^2}{2m}\Psi^*_{xx}\Psi_x + V\Psi^*\Psi_x - \Big(-\frac{\hbar^2}{2m}\Psi^*\Psi_{xxx} + V_x|\Psi|^2 + V\Psi^*\Psi_x\Big)\Big]dx = \\
\frac{i}{\hbar}\int\Big[-\frac{\hbar^2}{2m}\Big(\Psi^*_{xx}\Psi_x - \Psi^*\Psi_{xxx}\Big) - V_x|\Psi|^2\Big]dx = -\frac{i}{\hbar}\int V_x|\Psi|^2 dx,
\end{gathered}$$

where the final equality can be demonstrated using integration by parts. (Assuming boundary contributions vanish, each integration by parts moves a derivative from one term to the other, and contributes a factor of -1.)

Thus, we conclude that

$$\boxed{\frac{d\langle p\rangle}{dt} = -\left\langle\frac{\partial V}{\partial x}\right\rangle = \langle F\rangle.}$$

This is known as **Ehrenfest's Theorem**. The two equations

$$\begin{aligned}
\frac{d\langle x\rangle}{dt} &= \frac{\langle p\rangle}{m} \\
\frac{d\langle p\rangle}{dt} &= \langle F\rangle
\end{aligned}$$

tell us that expectation values in quantum mechanics obey classical equations of motion.