Exercise/ We express the electrostatic energy in terms of the charge density and the electrostatic potential.

Source: Jackson, Chapter 1.

We start by rederiving the electrostatic potential.

From Gauss's law, we know that the electric field due to a point charge at position $\vec{r}'$ is given by

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

Then, by the principle of superposition, the electric field due to a collection of point charges is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_i q_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3} \approx \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} d^3r'$$

assuming the charge distribution is approximately continuous.

Next, we use the fact that $\nabla_{\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|} = -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$ to write

$$\vec{E}(\vec{r}) = -\nabla\left(\frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r'\right) \equiv -\nabla\Phi(\vec{r}).$$

Thus, we have derived the electrostatic potential

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r' \longleftrightarrow \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i q_i \frac{1}{|\vec{r}-\vec{r}_i|}$$

Next, we demonstrate the physical significance of the potential by computing the work required to move a charge q from $\vec{r}_0$ to $\vec{r}_1$ in the presence of an electric field $\vec{E}(\vec{r})$.

The work is given by

$$W = -\int_{\vec{r}_0}^{\vec{r}_1} \vec{F}\cdot d\vec{l} = -q\int_{\vec{r}_0}^{\vec{r}_1} \vec{E}(\vec{r})\cdot d\vec{l} = q\int_{\vec{r}_0}^{\vec{r}_1} \nabla\Phi(\vec{r})\cdot d\vec{l}$$

$= q\Phi(\vec{r}_1) - q\Phi(\vec{r}_0)$. The final equality comes from the fundamental theorem of line integrals. Thus, $q\Phi$ can be understood as the potential energy of a test charge in the electrostatic field.

Now, suppose we have a number of point charges $\{q_i, \vec{r}_i\}_i$. Then, the potential energy of a point charge j near this configuration will be given by

$$W_j = \sum_i q_j \Phi_i(\vec{r}_j) = \frac{1}{4\pi\epsilon_0}\sum_i q_j \frac{q_i}{|\vec{r}_j - \vec{r}_i|}$$

It follows that the total electrostatic energy of the configuration will be given by

$$W = \sum_j W_j = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

where the $1/2$ is included to avoid double-counting, as

$$\frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{q_j q_i}{|\vec{r}_j - \vec{r}_i|} .$$

Assuming our charge distribution can be approximated as a continuum, the electrostatic energy is given by

$$\boxed{W = \frac{1}{2} \cdot \frac{1}{4\pi\varepsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} \approx \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \iint \frac{\rho(\vec{r})\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3r' \, d^3r}$$

We can also observe that

$$\frac{1}{2} \frac{1}{4\pi\varepsilon_0} \iint \frac{\rho(\vec{r})\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3r' \, d^3r = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}) \left(\int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3r' \right) d^3r$$

$$\Longrightarrow \boxed{W = \frac{1}{2} \cdot \int \rho(\vec{r}) \Phi(\vec{r}) \, d^3r}$$