We first derive the equipartition theorem for a system with a quadric energy given by

Source: Schroeder, Intro to Thermal Physics

$$E(q) = cq^2,$$

Where q could be the position, momentum, etc. We suppose $q \in (-\infty, \infty)$ is continuous, and that our system is in equilibrium with a thermal bath at temperature T. We compute $\langle E \rangle$.

$$Z = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{-\beta c q^2} \, dq = \frac{1}{2\pi\hbar} \cdot \frac{1}{\sqrt{\beta c}} \int_{-\infty}^{\infty} e^{-\beta c q^2} \, d\sqrt{\beta c}\, q$$

$$= \frac{1}{2\pi\hbar} \cdot \frac{1}{\sqrt{\beta c}} \int_{-\infty}^{\infty} e^{-x^2} dx = \frac{1}{2\hbar} \frac{1}{\sqrt{\beta c \pi}}.$$

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} =$$

$$-2\hbar \sqrt{\beta c \pi} \cdot \frac{1}{2\hbar \sqrt{\pi c}} \cdot -\frac{1}{2} \beta^{-3/2} = \frac{1}{2\beta} = \frac{1}{2} k_B T.$$

This is the equipartition theorem for a system with a single quadratic degree of freedom: the average energy associated with that degree of freedom is $\frac{1}{2} k_B T$.

We next derive the equipartition theorem for system with a single linear degree of freedom.

Source: Schroeder, Intro Thermal Physics, Problem 6.31

Consider a system in equilibrium with a thermal bath at temperature T, with energy given by $E(q) = c|q|$, $q \in (-\infty, \infty)$. We compute its average energy:

$$Z = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\beta E(q)}\,dq = \frac{1}{\pi\hbar}\int_{0}^{\infty} e^{-\beta c q}\,dq =$$

$$\frac{1}{\pi\hbar\beta c}\int_0^\infty e^{-x}\,dx = \frac{1}{\pi\hbar\beta c}.$$

We can now compute the average energy

$$\langle E \rangle = -\frac{\partial}{\partial\beta}\ln Z = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = -\pi\hbar\beta c \cdot -\frac{1}{\pi\hbar c\beta^2}$$

$$= \frac{1}{\beta} = \boxed{k_B T}$$

Thus, the equipartition theorem for a system with a single linear degree of freedom is simply the statement that the average energy for that DOF is $k_B T$.

Next, consider a system where the energy is given by

$$E(q) = c|q|^n \quad \text{(we include the absolute value to ensure decay for large } q \text{ and odd } n.)$$

Its partition function is given by

$$Z = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\beta c|q|^n}\, dq = \frac{1}{2\pi\hbar(\beta c)^{1/n}}\int_{-\infty}^{\infty} e^{-|x|^n}\, dx$$

$$= \frac{2}{\pi\hbar(\beta c)^{1/n}}\,\frac{1}{n}\,\Gamma\left(\frac{1}{n}\right).$$

It follows that the average energy for this system is given by

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = -\frac{\pi\hbar(\beta c)^{1/n} n}{2\Gamma(1/n)}\cdot\frac{2\Gamma(1/n)}{\pi\hbar c^{1/n} n}\cdot\left(\frac{-1}{n}\right)\beta^{-1/n-1}$$

$$\frac{1}{\beta n} = \boxed{\frac{k_B T}{n}}$$

So more generally, the equipartition theorem for a single degree-n DOF states that the average energy for that DOF is $k_B T/n$.

Finally, consider a system with many degree-n DOFs whose energy is given by

$$E(q_1, \dots, q_m) = c_1|q_1|^n + \dots + c_m|q_m|^n$$

In this case, the partition function is given by

$$Z = \frac{1}{(2\pi\hbar)^m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dq_1 \cdots dq_m \, e^{-\beta(c_1|q_1|^n + \dots + c_m|q_m|^n)}$$

$$= \frac{1}{(2\pi\hbar)^m} \left(\int_{-\infty}^{\infty} e^{-\beta c_1|q_1|^n} dq_1 \right) \cdot \ldots \cdot \left(\int_{-\infty}^{\infty} e^{-\beta c_m|q_m|^n} dq_m \right)$$

$$= \frac{1}{(2\pi\hbar)^m} \cdot \frac{1}{\beta^{m/n}} \cdot \frac{2^m}{(c_1 \cdot \dots \cdot c_m)^{1/n}} \frac{1}{n^m} \Gamma\left(\frac{1}{n}\right)^m$$, which implies

that

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} =$$

$$-(c_1 \cdots c_m)^{1/n} \left(\frac{2\pi\hbar \beta^{1/n} n}{2\Gamma(1/n)} \right)^m \left(\frac{2\Gamma(1/n)}{2\pi\hbar \, n} \right)^m \frac{1}{(c_1 \cdots c_m)^{1/n}} (-m/n) \beta^{-m/n - 1}$$

$$= \frac{m}{n\beta} = \boxed{m \cdot \frac{k_B T}{n}}$$

Thus, we arrive at a more general version of the equipartition theorem: For a system with m degree-n DOFs,

$$\boxed{\frac{\langle E \rangle}{m} = \frac{k_B T}{n}}.$$

That is, the average energy per DOF is $k_B T/n$, and equivalently, the average total energy is $m \cdot k_B T/n$.