# Evolution of Two Point Particles Under Coulomb Force in 1D

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**Exercise:**

We derive the equation of motion governing the evolution of two point particles interacting via the Coulomb force in one dimension.

**Solution:**

Suppose we have two point particles with masses $m_1$ and $m_2$, charges $q_1$ and $q_2$, and positions $x_1$ and $x_2$ which interact only via the Coulomb force. The energy of this system is given by

$$E = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{kq_1q_2}{|x_1 - x_2|}.$$

If these quantities are given in the center-of-mass frame, we have

$$\dot{x}_{\text{COM}} = 0 = \frac{m_1\dot{x}_1 + m_2\dot{x}_2}{m1 + m2} \implies \dot{x}_1 = -\frac{m_2}{m_1}\dot{x}_2.$$

Substituting into the equation for energy conservation, we have

$$E = \frac{1}{2}m_1\Big(-\frac{m_2}{m_1}\dot{x}_2\Big)^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{kq_1q_2}{|x_1 - x_2|} = \frac{1}{2}m_2\Big(1 + \frac{m_2}{m_1}\Big)\dot{x}_2^2 + \frac{kq_1q_2}{|x_1 - x_2|}.$$

Now, define $x \equiv x_1 - x_2$. It follows that

$$\dot{x} = \dot{x}_1 - \dot{x}_2 = -\frac{m_2}{m_1}\dot{x}_2 - \dot{x}_2 = -\Big(1 + \frac{m_2}{m_1}\Big)\dot{x}_2 \implies \dot{x}^2 = \Big(1 + \frac{m_2}{m_1}\Big)^2\dot{x}_2^2 \implies$$
$$E = \frac{1}{2}\Big(\frac{m_2}{1 + \frac{m_2}{m_1}}\Big)\dot{x}^2 + \frac{kq_1q_2}{|x|} = \frac{1}{2}\Big(\frac{m_1m_2}{m_1 + m_2}\Big)\dot{x}^2 + \frac{kq_1q_2}{|x|} \equiv \frac{1}{2}\mu\dot{x}^2 + \frac{kq_1q_2}{|x|},$$

where $\mu$ is the reduced mass. Solving for $\dot{x}$, we derive our equation of motion,

$$\boxed{\dot{x} = \sqrt{\frac{2}{\mu}\Big(E - \frac{kq_1q_2}{x}\Big)}.}$$

Note that we may always define $x_1 > x_2$ and so we drop the absolute value. To solve this differential equation, we compute $E$ at $t = 0$, and use the initial condition $\dot{x}(0) = \dot{x}_1(0) - \dot{x}_2(0)$.

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From this point, there are two ways to proceed. This differential equation can be solved numerically, yielding an approximate trajectory $x(t)$. Alternatively, we can solve the differential equation analytically using Mathematica, and we get

$$\frac{dx}{dt} = f(x) \implies \int dt = \int \frac{dx}{f(x)} \implies$$

$$t(x) + C = \frac{\sqrt{E}(xE - kq_1q_2) + kq_1q_2\sqrt{E - \frac{kq_1q_2}{x}}\tanh^{-1}\left(\frac{\sqrt{E - \frac{kq_1q_2}{x}}}{\sqrt{E}}\right)}{\sqrt{2}E^{3/2}\sqrt{\frac{xE - kq_1q_2}{\mu x}}} \equiv$$

$$\frac{\sqrt{E}x\alpha(x) + kq_1q_2\sqrt{\alpha(x)}\tanh^{-1}\sqrt{\frac{\alpha(x)}{E}}}{\sqrt{\frac{2}{\mu}}E^{3/2}\sqrt{\alpha(x)}} = \sqrt{\frac{\mu}{2}}\Big(\frac{x\sqrt{\alpha(x)}}{E} + \frac{kq_1q_2}{E^{3/2}}\tanh^{-1}\sqrt{\frac{\alpha(x)}{E}}\Big) \implies$$

$$\boxed{t(x) + C = \sqrt{\frac{\mu}{2E}}\Big(x\sqrt{\frac{\alpha(x)}{E}} + \frac{kq_1q_2}{E}\tanh^{-1}\sqrt{\frac{\alpha(x)}{E}}\Big),}$$

where $\alpha(x) \equiv E - \frac{kq_1q_2}{x}$. This expression cannot be inverted analytically to get $x(t)$, but it may be numerically inverted at a finite list of $x$ values, and in that way a discretized but exact trajectory can be reconstructed.