Exercise/ We derive the first-order energy correction in nondegenerate perturbation theory.

(Source: Griffiths 7.1).

Suppose we have some unperturbed system which we can solve exactly:

$$H^0|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle, \quad \langle\psi_m^0|\psi_n^0\rangle = \delta_{nm}.$$

Next, we introduce a perturbed Hamiltonian which we cannot solve exactly:

$$H = H^0 + \lambda H'.$$

We Taylor expand the states and energies

$$|\psi_n\rangle \approx |\psi_n^0\rangle + \lambda|\psi_n'\rangle$$

$$E_n \approx E_n^0 + \lambda E_n',$$

and then we can expand the perturbed Schrödinger equation

$$H|\psi_n\rangle = E_n|\psi_n\rangle \Rightarrow (H^0 + \lambda H')(|\psi_n^0\rangle + \lambda|\psi_n'\rangle)$$

$$= (E_n^0 + \lambda E_n')(|\psi_n^0\rangle + \lambda|\psi_n'\rangle) \Rightarrow H^0|\psi_n^0\rangle + \lambda H^0|\psi_n'\rangle$$

$$+\lambda H^1|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle + \lambda E_n^0|\psi_n^1\rangle + \lambda E_n^1|\psi_n^0\rangle,$$

where we have only kept terms linear in λ. Grouping terms by λ, we have two equations:

$H^0|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle$, our unperturbed system, and

$$H^0|\psi_n^1\rangle + H^1|\psi_n^0\rangle = E_n^0|\psi_n^1\rangle + E_n^1|\psi_n^0\rangle \Longrightarrow$$

$$\langle\psi_n^0|H^0|\psi_n^1\rangle + \langle\psi_n^0|H^1|\psi_n^0\rangle = E_n^0\langle\psi_n^0|\psi_n^1\rangle +$$

$$E_n^1\langle\psi_n^0|\psi_n^0\rangle \Longrightarrow E_n^0\langle\psi_n^0|\psi_n^1\rangle + \langle\psi_n^0|H^1|\psi_n^0\rangle$$

$$= E_n^0\langle\psi_n^0|\psi_n^1\rangle + E_n^1 \Longrightarrow$$

$$\boxed{E_n^1 = \langle\psi_n^0|H^1|\psi_n^0\rangle}$$

That is, the first correction to the n^{th} energy level is given by the expectation of the perturbation in the n^{th} unperturbed state.