Exercise

We show that the Fourier transform of a gaussian is a gaussian.

$$\mathcal{F}\left[e^{-(x-\mu)^2/2\sigma^2}\right] = \int_{-\infty}^{\infty} e^{-ikx} e^{-(x-\mu)^2/2\sigma^2} dx .$$

We complete the square in the exponent by finding a, b, c such that

$$a(x-b)^2 + c = ikx + (x-\mu)^2/2\sigma^2$$

We start by equating their second derivatives:

$$2a = \frac{1}{\sigma^2} \implies a = \frac{1}{2\sigma^2}$$

Next, we equate their first derivatives

$$\frac{1}{\sigma^2}(x-b) = ik + \frac{1}{\sigma^2}(x-\mu) \implies b = \mu - ik\sigma^2$$

Substituting this as well, we have

$$\frac{1}{2\sigma^2}\left(x - (\mu - ik\sigma^2)\right)^2 + c = ikx + (x-\mu)^2/2\sigma^2 \implies$$

$$\left(x^2 - 2x(\mu - ik\sigma^2) + (\mu - ik\sigma^2)^2\right) \cdot \frac{1}{2\sigma^2} + c = ikx +$$

$$\frac{1}{2\sigma^2}(x^2-2x\mu+\mu^2) \Rightarrow (-2i\mu k\sigma^2 - k^2\sigma^4)\cdot\frac{1}{2\sigma^2} + c = 0$$

$$\Rightarrow c = \frac{k^2\sigma^2}{2} + i\mu k.$$

Now, our fourier transform can be expressed as

$$\int_{-\infty}^{\infty} e^{-ikx} e^{-(x-\mu)^2/2\sigma^2} dx = e^{-\frac{k^2\sigma^2}{2}} e^{-i\mu k} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-(\mu-ik\sigma^2))^2} dx$$

$$= e^{-\frac{k^2\sigma^2}{2}} e^{-i\mu k} \int_{-\infty}^{\infty} e^{-u^2/2\sigma^2} du = \boxed{\sqrt{2\pi\sigma^2}\, e^{-k^2\sigma^2/2} e^{-i\mu k}}$$

We see that the Fourier transform of a gaussian will be a gaussian, so long as the initial gaussian has mean 0.

We can complete the square again to see if things can become clearer with the nonzero offset:

$$a(k-b)^2 + c = \frac{\sigma^2}{2}\cdot k^2 + i\mu k$$

Equating second derivatives, we see that

$$2a = \sigma^2 \Rightarrow a = \sigma^2/2.$$

Equating first derivatives, we have

$$\sigma^2(k-b) = \sigma^2 k + i\mu \implies b = -i\mu/\sigma^2.$$

Finally, comparing the initial equations, we have

$$\frac{\sigma^2}{2}\left(k + i\mu/\sigma^2\right)^2 + c = \frac{\sigma^2}{2}\left(k^2 + 2i\mu k/\sigma^2 - \frac{\mu^2}{\sigma^4}\right) + c$$

$$= \frac{\sigma^2 k^2}{2} + i\mu k \implies -\frac{\mu^2}{2\sigma^2} + c = 0 \implies c = \frac{\mu^2}{2\sigma^2}.$$

Thus, the Fourier transform of our shifted gaussian obeys

$$\boxed{\mathcal{F}\left[e^{-(x-\mu)^2/2\sigma^2}\right] = \sqrt{2\pi\sigma^2}\, e^{-\mu^2/2\sigma^2} e^{-\frac{\sigma^2}{2}(k+i\mu/\sigma^2)^2}}$$

So indeed, the Fourier transform of an offset gaussian is also a gaussian, just with an <u>imaginary</u> offset in k-space!