We compute the gradient in spherical coordinates.

In spherical coordinates, we have

$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$

We will assume that we already know the spherical unit vectors

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

The gradient is given by

$$\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z} =$$

$$\hat{r}\, g_r + \hat{\theta}\, g_\theta + \hat{\phi}\, g_\phi.$$

We find g_r, g_θ, g_ϕ.

$$\hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z} = \hat{x}\left(f_r r_x + f_\theta \theta_x + f_\phi \phi_x\right)$$

$+\hat{y}\left(f_r r_y + f_\theta \theta_y + f_\phi \phi_y\right) + \hat{z}\left(f_r r_z + f_\theta \theta_z + f_\phi \phi_z\right).$

Inverting our spherical coordinates, we have

$r = \sqrt{x^2+y^2+z^2}$, $\theta = \arccos(z/\sqrt{x^2+y^2+z^2})$, $\phi = \arctan(y/x)$.

The derivatives can be computed simply in Mathematica. Plugging in, we have

$$\vec{\nabla} f = \hat{x}\left(f_r \sin\theta\cos\phi + f_\theta \cos\theta\cos\phi/r - f_\phi \sin\phi/r\sin\theta\right) +$$

$$\hat{y}\left(f_r \sin\theta\sin\phi + f_\theta \cos\theta\sin\phi/r + f_\phi \cos\phi/r\sin\theta\right) +$$

$$\hat{z}\left(f_r \cos\theta - f_\theta \sin\theta/r\right) =$$

$$\left(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}\right) f_r +$$

$$\left(\cos\theta\cos\phi/r\,\hat{x} + \cos\theta\sin\phi/r\,\hat{y} - \sin\theta/r\,\hat{z}\right) f_\theta +$$

$$\left(-\sin\phi/r\sin\theta\,\hat{x} + \cos\phi/r\sin\theta\,\hat{y}\right) f_\phi =$$

$$\boxed{\frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi} = \vec{\nabla} f}$$