

We show that Hermitian matrices have real eigenvalues.

Suppose $Q = Q^\dagger$. Let $|\lambda\rangle$ be an eigenvector of Q with eigenvalue λ . Then

$$Q|\lambda\rangle = \lambda|\lambda\rangle \Rightarrow \langle\lambda|Q^\dagger = \langle\lambda|\lambda^* = \langle\lambda|Q.$$

It follows that

$$\langle\lambda|Q|\lambda\rangle = \langle\lambda|\lambda|\lambda\rangle = \langle\lambda|\lambda^*|\lambda\rangle$$

$$\Rightarrow \lambda = \lambda^*, \text{ assuming we have orthonormalized the eigenvectors} \\ \text{(for example, by using Gram-Schmidt).}$$

Thus, the eigenvalues of Hermitian matrices are real.