1D Infinite Square Well Eigenstates Are Orthonormal

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Exercise:

We demonstrate that the 1D infinite square well eigenstates are orthonormal.

Solution:

We consider the potential

$$V(x) = \begin{cases} 0 & 0 < x < a \\ \infty & otherwise \end{cases}$$

The eigenstates are given by

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right), \quad n \in \mathbb{N}.$$

We verify the orthonormality of these states.

$$\begin{gathered}
\langle\psi_n|\psi_m\rangle = \int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\sqrt{\frac{2}{a}}\sin\left(\frac{m\pi x}{a}\right)dx = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx = \\
\frac{1}{a}\int_0^a \left[\cos\left(\frac{(n-m)\pi x}{a}\right) - \cos\left(\frac{(n+m)\pi x}{a}\right)\right]dx = \\
\frac{1}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{a}\right)\Big|_0^a - \frac{1}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{a}\right)\Big|_0^a = \\
\frac{1}{(n-m)\pi}\sin\Big((n-m)\pi\Big) = \delta_{nm} \implies \\
\boxed{\langle\psi_n|\psi_m\rangle = \delta_{nm}.}
\end{gathered}$$