Exercise :

We derive the Klein - Gordon equation.

Solution :

Consider a Lagrangian for a field which contains a kinetic term and
a mass term.

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 .$$

The equations of motion for this field are given by the Euler-Lagrange
equations :

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = -m^2 \phi - \partial_\mu (\partial_\mu \phi)$$

$$\Longrightarrow \boxed{0 = (\partial^\mu \partial_\mu - m^2) \phi}$$

This is the Klein - Gordon equation.