Exercise: We derive the normalization and MLE for a powerlaw.

(Source: Power Law Distributions in Empirical ~~Data~~, Clauset et. al.)

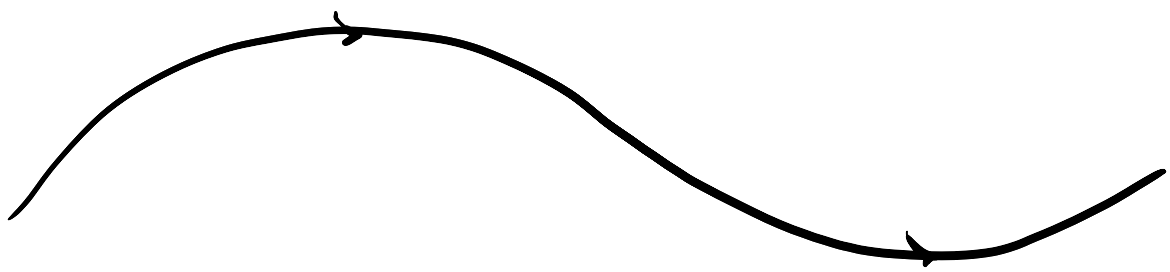
We consider a probability density $\rho(x) = Ax^{-\alpha}$, $x \in [x_0, \infty)$ and $\alpha > 1$. The normalization constant can be found according to

$$\int_{x_0}^{\infty} \rho(x)dx = 1 \Rightarrow$$

$$\frac{1}{A} = \int_{x_0}^{\infty} x^{-\alpha} dx = \frac{1}{1-\alpha} x^{1-\alpha} \Big|_{x_0}^{\infty} =$$

$$\frac{1}{\alpha - 1} x_0^{1-\alpha} \Rightarrow A = \frac{(\alpha-1)}{x_0} x_0^{-\alpha} \Rightarrow$$

$$\boxed{p(x) = \frac{\alpha-1}{x_0}\left(\frac{x}{x_0}\right)^{-\alpha}}$$



We now derive the maximum likelihood estimation (MLE) for the parameter α.

Suppose we have some data $\{x_i\}_{i=1}^{n}$ sampled i.i.d. from p. The likelihood of that dataset is given by

$$p(x_1, \dots, x_n \mid \alpha) = \prod_{i=1}^{n} p(x_i \mid \alpha),$$

So the log-likelihood is given by

$$\log p(x_1, \dots, x_n \mid \alpha) = \sum_{i=1}^{n} \log p(x_i \mid \alpha) =$$

$$\sum_{i=1}^{n} \log\left[\frac{\alpha-1}{x_0}\left(\frac{x_i}{x_0}\right)^{-\alpha}\right] = \sum_{i=1}^{n}\left[\log(\alpha-1) - \log x_0 - \alpha \log x_i/x_0\right]$$

$$= n\log(\alpha-1) - \log x_0 - \alpha \sum_{i=1}^{n} \log x_i/x_0 .$$

$$\alpha_{MLE} \equiv \hat{\alpha} \equiv \underset{\alpha}{\arg\max} \log p(x_1, \dots, x_n | \alpha),$$

Which we can find by differentiation

$$\frac{\partial \log p(x_1, \dots, x_n|\alpha)}{\partial \alpha} = 0 = \frac{n}{\alpha-1} - \sum_{i=1}^{n} \log x_i/x_0$$

$$\Rightarrow \frac{\alpha-1}{n} = \left(\sum_{i=1}^{n} \log x_i/x_0\right)^{-1} \Rightarrow$$

$$\boxed{\hat{\alpha} = 1 + n\left(\sum_{i=1}^{n} \log x_i/x_0\right)^{-1}}$$