We derive the existence of a phase transition in a model proposed by Lev Landau.

We wish to describe a system which exhibits a phase transition to a state in which many spins point in the same direction. For simplicity, let each spin only point up or down: $s_i \in \{-1, 1\}$. This system's magnetization is given by $M = \sum_i s_i$.

We might describe such a system's free energy using a series expansion in the magnetization.

$$F(M) \approx F_0 + a(T)M^2 + bM^4$$

where we have excluded odd powers of M because the energy of the system does not depend on the sign of M, but rather its magnitude. We assume $b > 0$ for simplicity.

Now, suppose there exists some temperature T_c at which $a(T)$ changes sign. The simplest case is that $a(T) = a_0(T - T_c)$ for some $a_0 > 0$. The equilibrium point of this system occurs when

$$\frac{\partial F}{\partial M} = 0 = 2M\left(a_0(T - T_c) + 2bM^2\right) \implies$$

$$M = 0 \text{ or } M = \pm\sqrt{\frac{a_0(T_c - T)}{2b}}$$

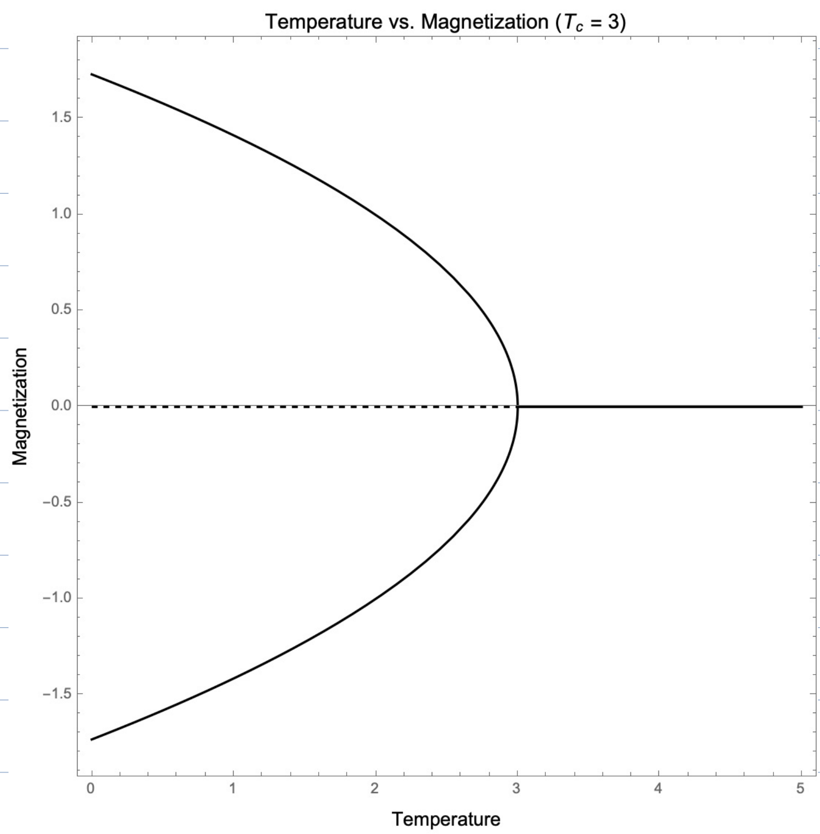
Evidently, this system has 3 equilibrium points, but only one is physical above T_c. We ask about the stability of each equilibrium point:

$$\frac{\partial^2 F}{\partial M^2} = 2a_0(T-T_c) + 12bM^2.$$

$$\left.\frac{\partial^2 F}{\partial M^2}\right|_{M=0} = 2a_0(T-T_c).$$ Thus, the zero magnetization state is stable for $T > T_c$, and becomes unstable when T is lowered below T_c.

$$\left.\frac{\partial^2 F}{\partial M^2}\right|_{M=\pm\sqrt{\frac{a_0(T_c-T)}{2b}}} = 2a_0(T-T_c) + \frac{12b\,a_0(T_c-T)}{2b} = 4a_0(T_c-T).$$

So, by contrast, these two points are stable for $T < T_c$. Thus, as we vary the system's temperature through the critical point, its magnetization varies in this way:



where the solid lines indicate stable configurations, and the dashed lines indicate unstable configurations. This system exhibits a "(super critical) pitch fork bifurcation," which is to say a certain type of phase transition as the temperature is varied through the critical point. Below T_c, one of the two states of nonzero magnetization becomes thermodynamically favorable relative to the disordered state.