Exercise:

We derive the principle of least action for a field.

(Source: Tong Quantum Field Theory)

Solution:

Suppose we have a field governed by a Lagrangian density

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi).$$

Then the action is given by

$$S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi).$$

The action will be minimized when

$$\delta S = 0 \quad =$$

$$\delta \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi) \quad =$$

$$\int d^4x \, \delta \mathcal{L}(\phi, \partial_\mu \phi) \quad =$$

$$\int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \overset{\text{Integration by parts}}{=}$$

$$\int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \, .$$

Since $\delta\phi$ is arbitrary, we conclude that

$$\boxed{0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}$$

These are the Euler-Lagrange equations for the field.