Probability Conservation in Quantum Mechanics

Matt Kafker

Exercise:

We show that a wave function which is initially normalized preserves its normalization under time evolution.

Solution:

We start with a normalized wave function $\Psi(x,t)$, meaning it satisfies

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1.$$

In quantum mechanics, the wave function evolves according to the Schrödinger equation,

$$i\hbar\dot{\Psi}(x,t) = -\frac{\hbar^2}{2m}\Psi_{xx}(x,t) + V(x)\Psi(x,t),$$

where $V(x)$ is the potential, which we shall assume to be real for this exercise.

We examine the time-evolution of the normalization.

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi|^2 dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}|\Psi|^2 dx = \int_{-\infty}^{\infty} (\dot{\Psi}^*\Psi + \Psi^*\dot{\Psi}) dx.$$

From the Schrödinger equation, we have

$$\begin{aligned} \dot{\Psi} &= -\frac{i}{\hbar}\Big(-\frac{\hbar^2}{2m}\Psi_{xx} + V\Psi\Big) \\ \dot{\Psi}^* &= \frac{i}{\hbar}\Big(-\frac{\hbar^2}{2m}\Psi^*_{xx} + V\Psi^*\Big). \end{aligned}$$

Substituting, we have

$$\frac{i}{\hbar}\int_{-\infty}^{\infty}\left[-\frac{\hbar^2}{2m}\Psi^*_{xx}\Psi + V|\Psi|^2 - \left(-\frac{\hbar^2}{2m}\Psi^*\Psi_{xx} + V|\Psi|^2\right)\right]dx =$$
$$\frac{i}{\hbar}\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left(\Psi^*\Psi_{xx} - \Psi^*_{xx}\Psi\right)dx.$$

We can integrate the second term by parts twice:

$$\int_{-\infty}^{\infty}\Psi^*_{xx}\Psi dx = \Psi^*_x\Psi\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\Psi^*_x\Psi_x dx = -\left[\Psi^*\Psi_x\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\Psi^*\Psi_{xx}dx\right]$$
$$= \int_{-\infty}^{\infty}\Psi^*\Psi_{xx}dx.$$

However, this is simply the first term in the integrand above, so we conclude that

$$\boxed{\frac{d}{dt}\int_{-\infty}^{\infty}|\Psi|^2 dx = 0.}$$