We compute the expectation value of an arbitrary pairwise potential

$$V_{pair} = \sum_{\ell<m} V(\vec{x}_\ell, \vec{x}_m)$$

where $V(\vec{x}_\ell, \vec{x}_m) = V(\vec{x}_m, \vec{x}_\ell)$, for an N-particle system in a Slater determinant state:

$$\Psi = \frac{1}{\sqrt{N!}} \sum_{i_1,\dots,i_N=1}^{N} \varepsilon_{i_1\cdots i_N} \prod_{k=1}^{N} \phi_{i_k}(\vec{x}_k).$$

$$\langle V_{pair} \rangle = \int \prod_{k=1}^{N} d^3x_k \left(\frac{1}{\sqrt{N!}} \sum_{j_1\cdots j_N} \varepsilon_{j_1\cdots j_N} \phi^*_{j_k}(\vec{x}_k) \right)$$

$$\left(\sum_{\ell<m} V(\vec{x}_\ell, \vec{x}_m) \right) \left(\frac{1}{\sqrt{N!}} \sum_{i_1\cdots i_N} \varepsilon_{i_1\cdots i_N} \phi_{i_k}(\vec{x}_k) \right) =$$

$$\frac{1}{N!} \sum_{\substack{i_1\cdots i_N \\ j_1\cdots j_N}} \varepsilon_{j_1\cdots j_N} \varepsilon_{i_1\cdots i_N} \sum_{\ell<m} \int \prod_{k=1}^{N} d^3x_k \, \phi^*_{j_k}(\vec{x}_k) V(\vec{x}_\ell, \vec{x}_m) \phi_{i_k}(\vec{x}_k).$$

We now remark that, for all $i \neq j$, $V(\vec{x}_i, \vec{x}_j)$ appears in V_{pair}. Moreover, $\langle V_{pair} \rangle$ is invariant under arbitrary interchanges $\vec{x}_i \leftrightarrow \vec{x}_j$. It follows that each integral of $V(\vec{x}_i, \vec{x}_j)$ will have the same value, so we only perform one such integral. Without loss of generality, we choose $V(\vec{x}_1, \vec{x}_2)$. We multiply this result by the total number of terms in V_{pair}, $N(N-1)/2$. Thus, the previous line simplifies to

$$\frac{1}{N!}\sum_{\substack{i_1\cdots i_N\\ j_1\cdots j_N}} \varepsilon_{j_1\cdots j_N}\,\varepsilon_{i_1\cdots i_N}\ \frac{N(N-1)}{2}.$$

$$\left(\int d^3x_1 d^3x_2\, \phi^*_{j_1}(\vec{x}_1)\,\phi^*_{j_2}(\vec{x}_2)\,V(\vec{x}_1,\vec{x}_2)\,\phi_{i_1}(\vec{x}_1)\,\phi_{i_2}(\vec{x}_2)\right)\cdot$$

$$\left(\prod_{k=3}^{N}\int d^3x_k\, \phi^*_{j_k}(\vec{x}_k)\,\phi_{i_k}(\vec{x}_k)\right) =$$

$$\frac{1}{2(N-2)!}\sum_{\substack{i_1\cdots i_N\\ j_1\cdots j_N}} \varepsilon_{j_1\cdots j_N}\,\varepsilon_{i_1\cdots i_N}\left(\prod_{k=3}^{N}\delta_{i_k j_k}\right)\cdot$$

$$\left(\int d^3x_1 d^3x_2\, \phi^*_{j_1}(\vec{x}_1)\,\phi^*_{j_2}(\vec{x}_2)\,V(\vec{x}_1,\vec{x}_2)\,\phi_{i_1}(\vec{x}_1)\,\phi_{i_2}(\vec{x}_2)\right) =$$

$$\frac{1}{2(N-2)!}\sum_{\substack{i_1\cdots i_N\\ j_1, j_2}} \varepsilon_{j_1 j_2 i_3\cdots i_N}\,\varepsilon_{i_1 i_2\cdots i_N}\cdot$$

$$\left(\int d^3x_1 d^3x_2\, \phi^*_{j_1}(\vec{x}_1)\,\phi^*_{j_2}(\vec{x}_2)\,V(\vec{x}_1,\vec{x}_2)\,\phi_{i_1}(\vec{x}_1)\,\phi_{i_2}(\vec{x}_2)\right).$$

Now, suppose we fix i_1, i_2, j_1, j_2. The remaining permutations of $i_3, \ldots, i_N$ will contribute $(N-2)!$ copies of the same terms.

All that remains is to compute the sums over $i_1, i_2, j_1,$ and j_2. With $i_3, \ldots, i_N$ fixed, there are two cases: $(i_1 = j_1, i_2 = j_2)$ and $(i_1 = j_2, i_2 = j_1)$. Of course, $i_1 \neq i_2$ and $j_1 \neq j_2$ because of the Levi-Civita symbols. We treat each case separately.

Case 1: $i_1 = j_1$ and $i_2 = j_2$

$$\frac{1}{2(N-2)!} \sum_{\substack{i_1 \cdots i_N \\ j_1, j_2}} \varepsilon_{j_1 j_2 i_3 \cdots i_N} \varepsilon_{i_1 i_2 \cdots i_N} \cdot$$

$$\left(\int d^3x_1 d^3x_2 \, \phi^*_{j_1}(\vec{x}_1) \phi^*_{j_2}(\vec{x}_2) V(\vec{x}_1, \vec{x}_2) \phi_{i_1}(\vec{x}_1) \phi_{i_2}(\vec{x}_2) \right) =$$

$$\frac{1}{2(N-2)!} \sum_{i_1 \cdots i_N} \left(\varepsilon_{i_1 i_2 \cdots i_N} \right)^2 \cdot$$

$$\left(\int d^3x_1 d^3x_2 \, \phi^*_{i_1}(\vec{x}_1) \phi^*_{i_2}(\vec{x}_2) V(\vec{x}_1, \vec{x}_2) \phi_{i_1}(\vec{x}_1) \phi_{i_2}(\vec{x}_2) \right) =$$

$$\frac{1}{2(N-2)!} \cdot (N-2)! \sum_{i_1 \neq i_2} \left(\int d^3x_1 d^3x_2 \, \phi^*_{i_1}(\vec{x}_1) \phi^*_{i_2}(\vec{x}_2) V(\vec{x}_1, \vec{x}_2) \phi_{i_1}(\vec{x}_1) \phi_{i_2}(\vec{x}_2) \right)$$

$$= \sum_{i_1 < i_2} \int d^3x_1 d^3x_2 \, \phi^*_{i_1}(\vec{x}_1) \phi^*_{i_2}(\vec{x}_2) V(\vec{x}_1, \vec{x}_2) \phi_{i_1}(\vec{x}_1) \phi_{i_2}(\vec{x}_2),$$

where the final equality comes from the fact that $i_1 \longleftrightarrow i_2$ leaves the integral unchanged, and thus we have overcounted by a factor of two.

Case 2: $i_1 = j_2$ and $i_2 = j_1$

$$\frac{1}{2(N-2)!} \sum_{\substack{i_1 \cdots i_N \\ j_1, j_2}} \varepsilon_{j_1 j_2 i_3 \cdots i_N} \varepsilon_{i_1 i_2 \cdots i_N} \cdot$$

$$\left(\int d^3x_1 d^3x_2\, \phi^*_{j_1}(\vec{x}_1) \phi^*_{j_2}(\vec{x}_2) V(\vec{x}_1, \vec{x}_2) \phi_{i_1}(\vec{x}_1) \phi_{i_2}(\vec{x}_2) \right) =$$

$$\frac{1}{2(N-2)!} \sum_{i_1 \cdots i_N} \varepsilon_{i_2 i_1 i_3 \cdots i_N} \varepsilon_{i_1 i_2 \cdots i_N}$$

$$\left(\int d^3x_1 d^3x_2\, \phi^*_{i_2}(\vec{x}_1) \phi^*_{i_1}(\vec{x}_2) V(\vec{x}_1, \vec{x}_2) \phi_{i_1}(\vec{x}_1) \phi_{i_2}(\vec{x}_2) \right) =$$

$$\frac{1}{2(N-2)!} \cdot (N-2)! \sum_{i_1 \neq i_2} (-1) \cdot$$

$$\left(\int d^3x_1 d^3x_2\, \phi^*_{i_2}(\vec{x}_1) \phi^*_{i_1}(\vec{x}_2) V(\vec{x}_1, \vec{x}_2) \phi_{i_1}(\vec{x}_1) \phi_{i_2}(\vec{x}_2) \right) =$$

$$- \sum_{i_1 < i_2} \int d^3x_1 d^3x_2\, \phi^*_{i_2}(\vec{x}_1) \phi^*_{i_1}(\vec{x}_2) V(\vec{x}_1, \vec{x}_2) \phi_{i_1}(\vec{x}_1) \phi_{i_2}(\vec{x}_2) .$$

Finally, we add these two cases together (and clean up our variable names), and we get

$$\langle V_{pair} \rangle = \left\langle \sum_{\ell<m} V(\vec{x}_\ell, \vec{x}_m) \right\rangle =$$

$$\sum_{\ell<m} \int d^3x \, d^3y \left[\phi_\ell^*(\vec{x}) \phi_m^*(\vec{y}) V(\vec{x},\vec{y}) \phi_\ell(\vec{x}) \phi_m(\vec{y}) - \right.$$

$$\left. \phi_m^*(\vec{x}) \phi_\ell^*(\vec{y}) V(\vec{x},\vec{y}) \phi_\ell(\vec{x}) \phi_m(\vec{y}) \right] =$$

$$\boxed{\sum_{\ell<m} \langle \phi_\ell \phi_m | V | \phi_\ell \phi_m \rangle - \langle \phi_m \phi_\ell | V | \phi_\ell \phi_m \rangle}$$

if you define the matrix elements in a sensible way.