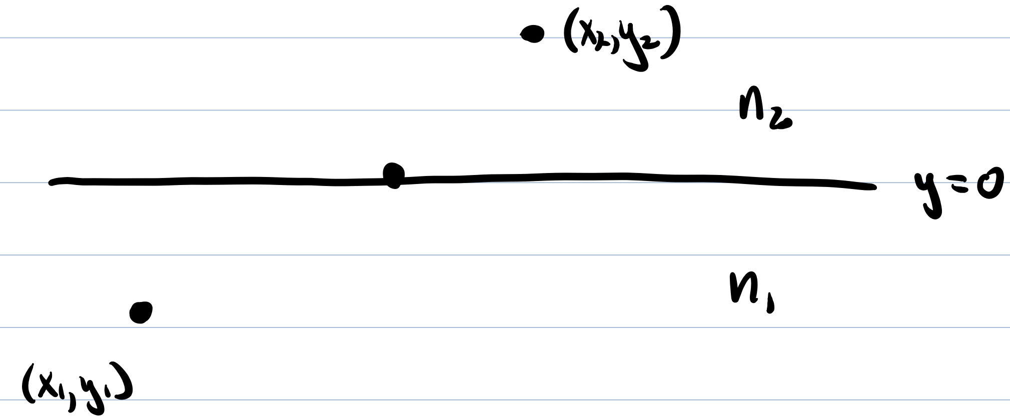
1.11

We derive Snell's law using the principle of least time.

We wish to find the trajectory which minimizes the time to go from (x_1, y_1) to (x_2, y_2) where there are two regions with two different indices of refraction:



The time can be expressed as an integral

$$T = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{ds}{v} = \frac{n_1}{c} \int_{(x_1,y_1)}^{(x^*,0)} ds + \frac{n_2}{c} \int_{(x^*,0)}^{(x_2,y_2)} ds$$

We also recall that $v = c/n$.

We first prove that the least-time trajectory in either medium is a straight line.

Let $f_1(x)$ be the curve that the particle follows from (x_1, y_1) to $(x^*, 0)$, such that $f_1(x_1) = y_1$, $f_1(x^*) = 0$

$$T[f_1] = \frac{n_1}{c}\int_{(x_1,y_1)}^{(x^*,0)} ds\,(f_1) = \frac{n_1}{c}\int_{x_1}^{x^*} \sqrt{1+f_1'(x')^2}\,dx' \Rightarrow$$

$$\frac{\delta T}{\delta f_1(\tilde{x})} = \frac{-d}{dx}\cdot\frac{f_1'(x)}{\sqrt{1+f_1'(x)^2}} = 0 \Rightarrow \frac{f_1'(x)}{\sqrt{1+f_1'(x)^2}} = \text{const} \equiv k$$

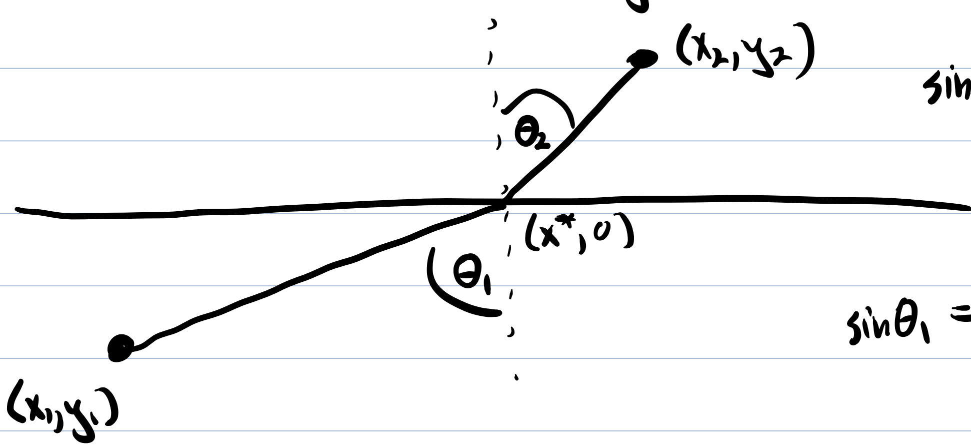
$$\Rightarrow [f_1'(x)]^2 = k^2(1+f_1'(x))^2 = k^2 + k^2 f_1'(x)^2 \Rightarrow$$

$$f_1'(x)^2(1-k^2) = k^2 \Rightarrow f_1'(x) = \sqrt{\frac{k^2}{1-k^2}} = \text{constant.}$$

Thus $f_1'(x) = \text{const} \Rightarrow f_1(x)$ is a straight line!

The same argument implies that $f_2(x)$ is a straight line.

We are now in a position to derive Snell's law. We wish to determine the position of x^* which minimizes the light's travel time.



$$\sin\theta_2 = \frac{x_2 - x^*}{\sqrt{(x_2-x^*)^2+y_2^2}}$$

$$\sin\theta_1 = \frac{x^* - x_1}{\sqrt{(x_1-x^*)^2+y_1^2}}$$

The time the light will travel is now given by

$$T = \frac{n_1}{c}\sqrt{(x_1 - x^*)^2 + y_1^2} + \frac{n_2}{c}\sqrt{(x_2 - x^*)^2 + y_2^2}$$

We find the minimum time by differentiating with respect to x^*:

$$\frac{dT}{dx^*} = 0 = \frac{n_1}{c} \cdot \frac{-(x_1 - x^*)}{\sqrt{(x_1 - x^*)^2 + y_1^2}} + \frac{n_2}{c} \frac{-(x_2 - x^*)}{\sqrt{(x_2 - x^*)^2 + y_2^2}} \Rightarrow$$

$$\frac{n_1 (x^* - x_1)}{\sqrt{(x_1 - x^*)^2 + y_1^2}} = \frac{n_2 (x_2 - x^*)}{\sqrt{(x_2 - x^*)^2 + y_2^2}}$$

However, we can see from the diagram that this expression is equivalent to

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

This is Snell's law.