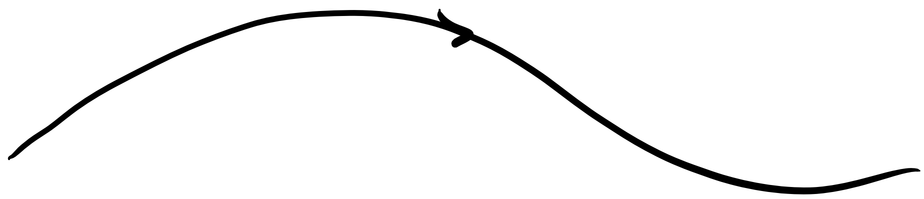
<u>Exercise</u>: We derive the virial theorem.

(Source: Hansen, Theory of Simple Liquids)



Let $\langle B \rangle_t \equiv \lim\limits_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau B(t)\,dt$.

Let us consider a collection of particles obeying Newton's laws:

$$m \ddot{\vec{r}}_i = \vec{F}_i, \quad i \in \{1, \dots, N\}.$$

We define the "<u>virial function</u>" as

$$\Phi(\{\vec{r}_j\}_j) = \sum_i \vec{r}_i \cdot \vec{F}_i .$$

It follows that its time-average is given by

$$\langle \Phi \rangle_t = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \sum_i \vec{r}_i(t) \cdot \vec{F}_i(t)\,dt =$$

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \sum_i \vec{r}_i(t)\cdot m\ddot{\vec{r}}_i(t)\,dt \overset{\text{IBP}}{=} \lim_{\tau\to\infty}\frac{1}{\tau}(-m)\int_0^\tau |\dot{\vec{r}}_i(t)|^2\,dt$$

$$= -2 \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \frac{|\vec{p}_i(t)|^2}{2m}\,dt = -2\langle K\rangle_t \Rightarrow$$

$$\boxed{\langle \Phi \rangle_t = -2\langle K\rangle_t}$$

This is the virial theorem.