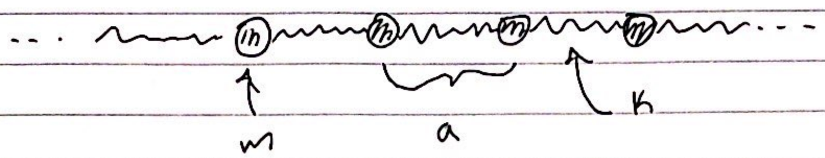
Exercise: We derive the wave equation.

Solution:

Suppose we have an array of masses "m" connected by springs in 1D:



where "a" is the separation between neighboring masses, and "K" is the spring constant. Let "x" denote the position coordinate along the chain, and let us consider the motion of the mass $u(x)$.

We show that $u(x)$ obeys the wave equation.

The force on $u(x)$ will be the net force due to its two neighbors:

$$F = +k\left[u(x+a) - u(x) + u(x-a) - u(x)\right] =$$

$$+K\left(u(x+a) - 2u(x) + u(x-a)\right) = +Ka^2 \frac{\left(u(x+a) - 2u(x) + u(x-a)\right)}{a^2} \approx$$

$+Ka^2 \dfrac{\partial^2 u}{\partial x^2}$. From Newton's second law, we know that

$$F = ma = m\frac{\partial^2 u}{\partial t^2} \Longrightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = \frac{Ka^2}{m}\frac{\partial^2 u}{\partial x^2}}$$

From dimensional analysis, we can see that Ka^2/m has units of $(\text{velocity})^2$, so we can also write this as

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}.$$