Wave Function Overlaps Are Preserved by Schrödinger Evolution

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Exercise:

We show that the Schrödinger equation preserves the overlap between two wave functions in time.

Solution:

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2 dx = \int_{-\infty}^{\infty}(\dot{\Psi}_1^*\Psi_2 + \Psi_1^*\dot{\Psi}_2)dx =$$
$$\int_{-\infty}^{\infty}\frac{i}{\hbar}\Big[-\frac{\hbar^2}{2m}\Psi_{1,xx}^*\Psi_2 + V\Psi_1^*\Psi_2 - \Big(-\frac{\hbar^2}{2m}\Psi_1^*\Psi_{2,xx} + V\Psi_1^*\Psi_2\Big)\Big]dx =$$
$$-\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\Big(\Psi_{1,xx}^*\Psi_2 - \Psi_1^*\Psi_{2,xx}\Big)dx = \boxed{0.}$$

The final equality can be obtained by integrating one of the terms by parts twice.

Of course, this implies that states which are initially orthogonal remain orthogonal under time-evolution.