Let A be an $N \times N$ matrix. We prove that $\det(A) = \det(A^T)$.

<u>Proof</u>

We recall the recursive definition of the determinant:

If the entries of A are denoted A_{ij}, then

$$\det A = \sum_i (-1)^{i+j} A_{ij} \det M_{ij} = \sum_j (-1)^{i+j} A_{ij} \det M_{ij}$$

where M_{ij} is the matrix A with the i^{th} row and j^{th} column removed.

We now prove the desired result inductively.

Base case: $N = 2$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \Rightarrow \det A = A_{11}A_{22} - A_{12}A_{21}$$

$$\|$$

$$A^T = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \Rightarrow \det A^T = A_{11}A_{22} - A_{21}A_{12} \quad .$$

Inductive Hypothesis: Suppose $\det B = \det B^T$ for all $(N-1) \times (N-1)$

matrices.

Let N_{ij} be A^T with the i^{th} row and j^{th} column removed.

$$A^T = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{N1} \\ A_{12} & A_{22} & \cdots & A_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1N} & A_{2N} & \cdots & A_{NN} \end{pmatrix}$$

Summing down the first column, we have

$$\det A^T = \sum_{j=1}^{N} A_{1j} (-1)^{1+j} \det N_{1j} =$$

$$\sum_{j=1}^{N} A_{1j} (-1)^{1+j} \det M_{j1} =$$

inductive hypothesis

$$\sum_{j=1}^{N} A_{1j} (-1)^{1+j} \det(M^T)_{1j} =$$

$$\sum_{j=1}^{N} A_{1j} (-1)^{1+j} \det M_{1j} = \det A$$

where the final equality can be understood by realizing that the previus expression is the recursive definition of the determinant of A if it is computed by summing across the first row:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix}$$

$\square$